Computations Using the Hatcher Spectral Sequence

Riley Moriss

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We have written about the setup and d^1 differential elsewhere. This spectral sequence computes

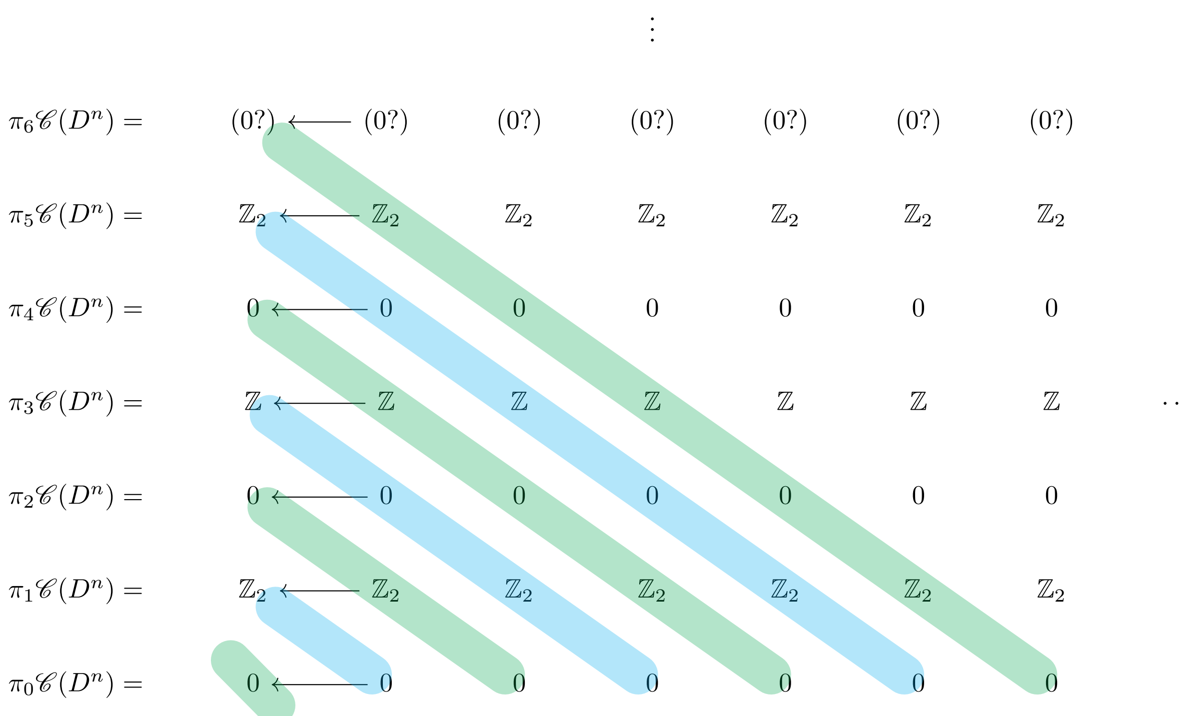
$$E^1_{pq} \implies \pi_{p+q+1}\frac{\widetilde{\mathrm{Diff}}_\partial(D^n)}{\mathrm{Diff}_\partial(D^n)}$$

and here we will try and use the known stable values of $\pi_i\mathscr{C}(D^n)$, which we have computed elsewhere, to get some concrete groups. Remember that the idiots way of thinking about the fact that a spectral sequence converges to some values, in this case $A_{p+q+1} = \pi_{p+q+1}\widetilde{\mathrm{Diff}}_\partial(D^n)/\mathrm{Diff}_\partial(D^n)$, means that the diagonal of the E^∞ page defined by a fixed n is given by some modules $E^\infty_{p,q}$ where $p+q+1 = n$ and that these modules form the composition factors of the module A_n. That is they describe a series of extension problems that give the group A_n. Here we summarise the calculations of $\pi_*\widetilde{\mathrm{Diff}}_\partial(D^n)/\mathrm{Diff}_\partial(D^n)$, noting that the value of n here is fixed but large.

<table>
<tr><th>*</th><th>Odd n</th><th>Even n</th></tr>
<tr><td>0</td><td colspan="2">0</td></tr>
<tr><td>1</td><td colspan="2">0</td></tr>
<tr><td>2</td><td colspan="2">$\mathbb{Z}_2$</td></tr>
<tr><td>3</td><td colspan="2">$\mathbb{Z}_2$</td></tr>
<tr><td>4</td><td>$\mathbb{Z}$ extended by $\mathbb{Z}_2$</td><td>?</td></tr>
<tr><td>5</td><td>$\mathbb{Z}_2$</td><td>?</td></tr>
<tr><td>6</td><td>A group of order $1, 2$ or 4</td><td>A group of order $2, 4$ or 8</td></tr>
</table>

1 What is possible

Lets look at the E^1 page in all its glory, noting that this sequence is first quadrant, and that is therefore all we will draw here. Note that (0?) denotes a conjecturally zero but currently unknown group (although it is known it has no small prime factors).



We have highlighted the diagonals that we will need to consider on the E^∞ page. The first green diagonal in the bottom left corner corresponds to $p = 0, q = 0 \implies p + q + 1 = 1$, the $p + q + 1 = 0$ diagonal is therefore ”off the page” in the bottom left consisting of all zeroes. We can see that the $p+q+1 = 7$ diagonal is the first to intersect the groups (0?) and hence until progress is made on these groups explicit computations using the Hatcher spectral sequence can only go up to $\pi_6\mathscr{C}(D^n)$ in the stable range. Beyond that something might be possible, certainly we could assume that these groups are zero and get conditional results.

2 The d^1 differential.

The first step in computing will be to see what is stable on the E^1 page and what continues to the later pages. This means computing the d^1 differential. Clearly all the zero groups are stable. Next we have that the differentials between the $\mathbb{Z}_2$'s are given by

$$\pi_i \mathscr{C}(D^{n+1}) = \mathbb{Z}_2 \to \mathbb{Z}_2 = \pi_i \mathscr{C}(D^n),$$
$$\sigma 1 = 1 \mapsto 1 + \bar{1} = 1 + 1 = 0$$

this is using three facts, the maps are determined by the generator, the suspension map which we assume is an iso therefore sends a generator to a generator $\sigma 1 = 1$ and finally that there is only one involution on $\mathbb{Z}_2$, given by the identity, hence $\bar{1} = 1$. Note that this was independent of n and independent of i so all the differentials on the E^1 page between any $\mathbb{Z}_2$ groups must be zero, in particular the differentials in the π_1, π_5 rows are zero. Hence all these $\mathbb{Z}_2$ groups persist to the later pages.

The only other non-trivial groups in this range are the $\mathbb{Z}$. For the same reason as before we have that this map is determined by where 1 maps and in particular it is known that it maps to $1 + \bar{1}$. Hence we have to determine the involution. First we can notice that there are only two involutions on the integers, those given by multiplication by ± 1. Following [Kup, 25.4] the involution on *rational K theory* is given by multiplication by -1 (on $K_i(\mathbb{Z})$ for $i > 0$) and moreover there is a commuting diagram (recalling that the groups in the right hand column are zero or $\mathbb{Q}$)

$$\begin{array}{ccc}
\pi_{i-2}\mathscr{C}(D^n) \otimes \mathbb{Q} & \xrightarrow{\sim} & K_i(\mathbb{Z}) \otimes \mathbb{Q} \\
\downarrow{\scriptstyle (-1)^{n+1}\overline{(-)}} & & \downarrow{\scriptstyle -1} \\
\pi_{i-2}\mathscr{C}(D^n) \otimes \mathbb{Q} & \xrightarrow{\sim} & K_i(\mathbb{Z}) \otimes \mathbb{Q}
\end{array}$$

But because the rational involution on $\pi_i \mathscr{C}(D^n)$ must come from the integral involution and in this case the integral groups are just the integers we also have a commuting integral diagram

$$\begin{array}{ccc}
\pi_{i-2}\mathscr{C}(D^n) & \xhookrightarrow{1 \mapsto 1} & K_i(\mathbb{Z}) \otimes \mathbb{Q} \\
\downarrow{\scriptstyle (-1)^{n+1}\overline{(-)}} & & \downarrow{\scriptstyle -1} \\
\pi_{i-2}\mathscr{C}(D^n) & \xleftarrow[\text{on the integers}]{n \mapsto n} & K_i(\mathbb{Z}) \otimes \mathbb{Q}
\end{array}$$

or if we take $x \in \pi_{i-2}\mathscr{C}(D^n)$ then we get that

$$-x = (-1)^{n+1}\bar{x}$$

or whats the same

$$\bar{x} = (-1)^n x.$$

Hence we have that the d^1 differential is given by

$$\pi_i \mathscr{C}(D^{n+1}) = \mathbb{Z} \to \mathbb{Z} = \pi_i \mathscr{C}(D^n),$$
$$1 \mapsto 1 + (-1)^n$$

Notice that this does depend on n, the dimension of the disc in the *codomain*, but it does not depend on i, that is if there is a row of $\mathbb{Z}$ then this is how the differential will act. In particular as we move to the right the differential will alternate, and we have to now deal with cases based on the parity of the dimension of the first disc.

The next concordance homotopy group is $\mathbb{Z} \oplus \mathbb{Z}_2$, because I have the involution on both peices, if I could show that it like splits accross the direct sum that would be nice, because then I could conjecturally go up further...

3 The Easy groups

3.1 π_0

As we pointed out the diagonal that converges to this group is outside the first quadrant and so all the groups are zero and so this group is zero.

3.2 π_1

The only group in this diagonal is the highlighted green zero and again this is stable so this group is zero.

3.3 π_2

The diagonal describing this group on E^1 is a $\mathbb{Z}_2$ in $(0,1)$ and a zero in $(1,0)$. The zero is stable. The map into the $\mathbb{Z}_2$ is zero as calculuated above and the map out is out of the first quadrant and hence also zero. Hence the $\mathbb{Z}_2$ survives into E^2. On E^2 however the map is comming from the row below which is all zeroes and on later pages we will be leaving the first quadrant and hence these are also zero. Thus this $\mathbb{Z}_2$ survives to the E^∞ page and we have stably that

$$\pi_2 \frac{\widetilde{\mathrm{Diff}_\partial}(D^n)}{\mathrm{Diff}_\partial(D^n)} \cong \mathbb{Z}_2$$

moreover there is a cokernel isomorphism

$$0 \to \pi_1 \mathscr{C}(D^n) \xrightarrow{\mathrm{coker}(0)} \pi_2 \frac{\widetilde{\mathrm{Diff}_\partial}(D^n)}{\mathrm{Diff}_\partial(D^n)}.$$

3.4 π_3

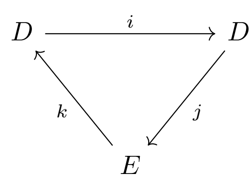
It is clear that this group is also $\mathbb{Z}_2$ for essentially the same reason as the previous group. The diagonal has only an extra 0 in the $(0,2)$ position and the $\mathbb{Z}_2$ and 0 have shifted in the p coordinates. The d^1 differentials are still all zero by our general arguments and the later differentials are all easily seen to be either coming from or going to zeroes and hence the $\mathbb{Z}_2$ in $(1,1)$ is stable and isomorphic to this group. We again get a cokernel isomorphism

$$0 \to \pi_1 \mathscr{C}(D^{n+1}) \xrightarrow{\mathrm{coker}(0)} \pi_3 \frac{\widetilde{\mathrm{Diff}_\partial}(D^n)}{\mathrm{Diff}_\partial(D^n)}.$$

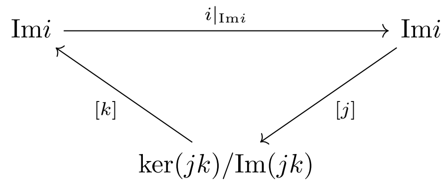
but note the shift in the dimension of the disc.

4 The d^3 differential

The next groups would involve the non-trivial d^1 maps and hence we would need to then check the subsequent differentials. Things here are begining to get messy but not incomprehensible. First recall that if our spectral sequence is defined by an exact couple



then the subsequent pages are given by the derived couple, which is given by the exact couple



$[k]$ just acts on a representative of a coset, and $[j]$ chooses a preimage under i and then goes through the quotient. Thus putting in degrees and following [Wei94] we have that the d^r differential is given by

$$E^r_{pq} \xrightarrow{k} D^r_{p-1,q} \xrightarrow{i} D^r_{p,q-1} \xrightarrow{j^{(r)}} E_{p-r,q+r-1}$$

Because i has degree $(1,-1)$ we get that $D^r_{p,q}$ is a submodule of $D^0_{p-r,q+r}$. Now by inspecting the construction of the HSS and enforcing that k has degree $(-1,0)$ and j has degree $(0,0)$ (because we are on E^1, we get that

$$D^0_{pq} = \begin{cases} \pi_q \widetilde{A}(D^{n+p+1}), & q \geq 0, p \geq -1 \\ \pi_{p+q} \frac{\widetilde{\mathrm{Diff}_\partial(D^n)}}{\mathrm{Diff}_\partial(D^n)}, & q < 0, p \geq -q \\ 0, & \text{else.} \end{cases}$$

where the $q < 0$ case is as [Kup] notes to fix the fact that the fibrations are not exact at the very end. Now we can start to think about the higher differentials.

Consider $q \geq 1$ then we have the differentials

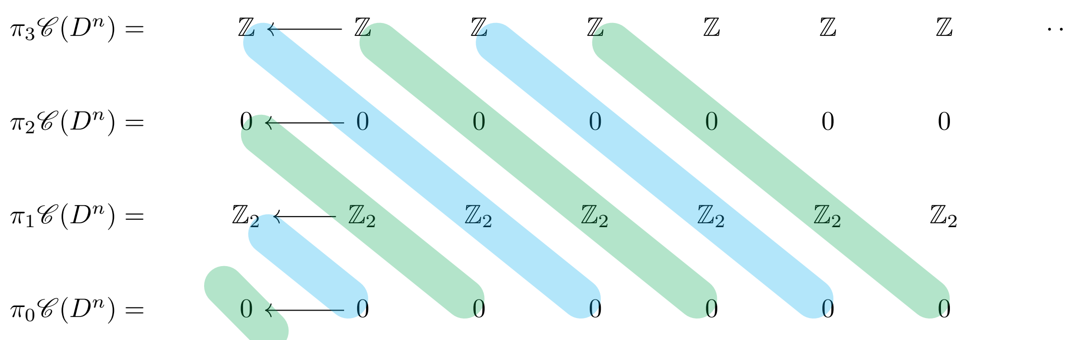
$$E^r_{pq} \xrightarrow{k} D^r_{p-1,q} \subseteq D^0_{p-1-r,q+r} \xrightarrow{i} D^r_{p,q-1} \subseteq D^0_{p-r,q-1+r} \xrightarrow{j^{(r)}} E_{p-r,q+r-1}$$

but since $r \geq 0, q \geq 1$ we see that $D^0_{p-1-r,q+r} = 0$ when $p < r$ and $D^0_{p-r,q-1+r} = 0$ when $p < r-1$. So we have shown that

Lemma. *If $p < r$ then $d^r_{p,q}$ is zero.*

Proof. Note that when $q = 0$ then this map is zero because the $E^r_{p,0}$ are zero.

5 π_4



The relevant diagonal is the second blue diagonal. We have already computed the d^1 differentials and see that we have to make an argument based on the parity of n.

5.1 n odd

Then by our formula the differential in to the $\mathbb{Z}$ in $(0,3)$ is trivial, that is the zero map. Hence this group survives to E^2 and because the row below is zero it must also survive to E^3. Now on E^3 it is possible for it to recieve a differential from the $\mathbb{Z}_2$ in position $(3,1)$, which survives also because its maps on E^1 are all zero and the row above is all zero so it survives d^2 too. There are however no non-zero group homorphisms from $\mathbb{Z}_2 \to \mathbb{Z}$ and so the $\mathbb{Z}$ survives E^3 and hence to E^∞. Thus we have an extension problem

$$0 \to \mathbb{Z} \to \pi_4 \frac{\widetilde{\mathrm{Diff}}_\partial(D^n)}{\mathrm{Diff}_\partial(D^n)} \to \mathbb{Z}_2 \to 0$$

5.2 n even

If n is even then there is a non-trivial d^1 into the $\mathbb{Z}$ at $(0,3)$, with image clearly $2\mathbb{Z}$. Thus on the E^2 and therefore E^3 page we have a $\mathbb{Z}_2$ which recieves a map from the $\mathbb{Z}_2$ in $(3,1)$. So we need to determine the $d^3_{3,1}$. potentially non-trivial.

6 π_5

Because the next row above is also full of zeroes the calculation is similar. Now however because we are interested in the $\mathbb{Z}$ in position $(1,3)$ we have to look at the parity of $n+1$, hence we have sort of the opposite convention to the above, however there is also another $\mathbb{Z}$ to consider. In detail:

6.1 n odd

In this case the differential on E^1 is trivial coming in and non-trivial going out. The non-trivial map out is an injection and so its trivial kernel is the thing that survives to E^∞. Hence we have that

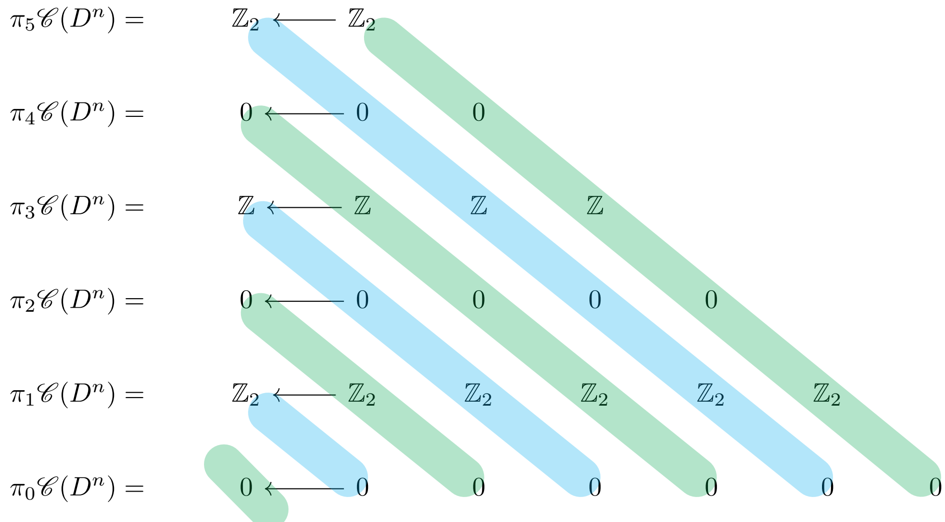
$$\pi_5 \frac{\widetilde{\mathrm{Diff}}_\partial(D^n)}{\mathrm{Diff}_\partial(D^n)} \cong \mathbb{Z}_2 \cong \pi_1 \mathscr{C}(D^{n+4})$$

6.2 n even

The d^1 coming in to the relevant $\mathbb{Z}$ is non-trivial and hence out is trivial. Thus we get a $\mathbb{Z}_2$ on E^3. The differential on E^3 going out will be zero but again it may receive a non-trivial map from the $\mathbb{Z}_2$ in the $(4,1)$. Thus the differential to compute would be $d^3_{4,1}$.

7 π_6

The section of the E^1 page that we consider is now



Restricting attention to the non-zero groups in the third blue diagonal we have the following E^3 page, where the groups are basically computed by the previous sections

$$0 \xleftarrow{\quad d^1, d^3 \quad} \pi_5\mathscr{C}(D^n) = \mathbb{Z}_2 \xleftarrow{\quad d^3 = ? \quad} \pi_3\mathscr{C}(D^{n+3}) \supseteq E^3 = \begin{cases} \mathbb{Z}_2, & \text{n odd} \\ 0, & \text{n even} \end{cases}$$

$$0 \xleftarrow{\quad d^3 \quad} \pi_3\mathscr{C}(D^{n+2}) \supseteq E^3 = \begin{cases} \mathbb{Z}_2, & \text{n even} \\ 0, & \text{n odd} \end{cases} \xleftarrow{\quad d^3 = ? \quad} \pi_1\mathscr{C}(D^{n+6}) = \mathbb{Z}_2$$

$$\pi_3\mathscr{C}(D^{n+1}) \supseteq E^3 = \begin{cases} \mathbb{Z}_2, & \text{n odd} \\ 0, & \text{n even} \end{cases} \xleftarrow{\quad d^3 \quad} \pi_1\mathscr{C}(D^{n+5}) = \mathbb{Z}_2 \xleftarrow{\quad d^1, d^3 \quad} 0$$

Thus we are interested in the following potentially non-zero differentials d^3_{pq} for the pq values $(3,3), (6,1), (5,1)$. Now lets do our case work.

7.1 n odd

When n is odd then we have a potentially non-trivial $d^3_{3,3}$, $d^3_{5,1}$, while $E^3_{2,3} = 0$. Moreover we would also need to consider the d^5 maps as there may be non-trivial maps between the $\mathbb{Z}_2$ from the π_1 row and the π_5 row. This does however "bound" the group to being smaller than a $\mathbb{Z}_2$ extended by a $\mathbb{Z}_2$. So it is one of the groups with order $1, 2, 4$, of which there are only four.

7.2 n even

In this case the only non-trivial differential relevant would be $d^3_{6,1}$, on the other hand the other two $\mathbb{Z}_2$ groups in the $1, 5$ rows will survive to E^5. The $\mathbb{Z}_2$ from the π_1 row is stable because its maps will leave the first quadrant. So without calculuating any more differentials we can only say that it has order either $2, 4, 8$.

Remark. If we assumed Van Divers conjecture it would be possible to go further. Particularly because the higher groups split and the differential is a homomotphisms and hence also splits we can decompose it into the differential on the pieces. In particular the $\pi_7\mathscr{C}$ groups are just a direct sum of $\mathbb{Z}_2 \oplus \mathbb{Z}$ and hence we have aready computed the differential on this group. The higher groups also sometimes have trivial involutions on them etc. It is unlikely to be able to attain exact results however so we drop it. It might be possible to give some bounds on the orders etc.

References

[Kup] Alexander Kupers. Lectures on diffeomorphism groups of manifolds, version February 22, 2019.

[Wei94] Charles A. Weibel. *An Introduction to Homological Algebra*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1994.